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Odd-Excited Binomial States of the Radiation Field and Some of Their Statistical Properties

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In this paper a new state called odd-excited binomial state (OEBS) is introduced. It interpolates between the odd number state and the odd-excited coherent state. We discuss some statistical properties, such as the Glauber second-order correlation function and squeezing phenomenon (normal and amplitude-squared squeezing) for this state. The quasiprobability distribution functions (Husimi Q-function and Wigner W-function) are also examined.

KEY WORDS: nonclassical states; squeezing; quasi-probability function.

1. INTRODUCTION

The most familiar state for the electromagnetic field is the Fock number state $|n\rangle$, which is an eigenstate of the photon number operator $\hat{n} = a^\dagger a$, i.e., $a^\dagger a|n\rangle = n|n\rangle$, where a and $a^\dagger$ are the annihilation and creation operators of the electromagnetic field, respectively. Fock states can be produced by applying the operator $a^{\dagger n}$ on the vacuum state $|0\rangle$ such that $|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle$. On the other hand, the coherent state $|\alpha\rangle$ can be obtained by applying the displacement operator $\hat{D}(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ on the vacuum state, where α is a complex amplitude.

Recently new quantum states have been introduced and investigated besides the number states and the coherent states. One of these states is the binomial state (BS) (Stoler *et al.*, 1985) which interpolates between the number state and the coherent state. Another state has been introduced to bridge between the thermal and the coherent states; it is the negative binomial state (NBS) (Agarwal, 1992; Joshi and Lawande, 1989, 1991). As for a further example, the generalized geometric state has been introduced to interpolate between the number state and the (nonpure)

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chaotic state (Batarfi *et al.*, 1995; Obada *et al.*, 1993, 1997). Furthermore, the even (odd) BS, interpolates between the even (odd) coherent state and the even (odd) number state (Abdalla *et al.*, 1994; Obada *et al.*, 1996).

Recently an excited binomial state (EBS) has been introduced and some of its properties have been discussed (Wang and Fu, 2000). These states interpolate between the Fock states and the excited coherent state (Wang and Fu, 2000).

In the present work we shall discuss the so-called odd-excited binomial states (OEBS) of the radiation field. They are introduced by repeated application of the photon creation operator on two BS. They may also be produced by superposition of two EBS. These states interpolate between the odd number state and the odd-excited coherent state. Therefore we shall devote the next section to the introduction of the OEBS and to the discussion of the statistical properties of the Glauber second-order correlation function. In Section 3 we shall consider the phenomenon of squeezing, especially the normal and the amplitude-squared squeezing. The quasipropability distribution function (W- and Q-functions) related to the OEBS are calculated in Section 4. Finally conclusions are drawn in Section 5.

2. THE OEBS

The EBS (see Wang and Fu, 2000) is defined as follows:

$$\begin{aligned}|k, \eta, M\rangle &= \lambda \sum_{n=0}^{M} \sqrt{\binom{M}{n}} \eta^n (1 - |\eta|^2)^{\frac{M-n}{2}} a^{\dagger k} |n\rangle \\ &= \lambda \sum_{n=0}^{M} C_n^M(k) |n + k\rangle \qquad (1)\end{aligned}$$

where λ is the normalization constant,

$$C_n^M(k) = \sqrt{\binom{M}{n}} \eta^n (1 - |\eta|^2)^{\frac{M-n}{2}} \sqrt{\frac{(n + k)!}{n!}}. \qquad (1a)$$

We introduce the OEBS through the definition

$$|k, \eta, M\rangle_{\mathrm{o}} = \lambda' \sum_{n=0} C_n^M(k) |n + k\rangle \qquad (2)$$

such that the state $|n + k\rangle$ is always odd. Therefore we have the following cases:

(i) for even k, i.e., $k = 2k_0$

$$|2k_0, \eta, M\rangle_{\mathrm{o}} = \lambda_0 \sum_{n=0}^{[\frac{M-1}{2}]} C_{2n+1}^M(2k_0) |2n + 2k_0 + 1\rangle \qquad (3a)$$

(ii) for odd k, i.e., $k = 2k_1 + 1$

$$|2k_1 + 1, \eta, M\rangle_{\mathrm{o}} = \lambda_0' \sum_{s=0}^{[\frac{M}{2}]} C_{2s}^{M}(2k_1 + 1)|2s + 2k_1 + 1\rangle \tag{3b}$$

where $C_{2n+1}^{M}(2k_0)$ and $C_{2s}^{M}(2k_1 + 1)$ are given from (1a), with $[\frac{N}{2}]$ being the largest integer less than or equal to $(\frac{N}{2})$. λ_0 and λ_0' are the normalization constants of OEBS. For k(even) and k(odd), respectively, given by

$$|\lambda_0|^{-2} = \sum_{n=0}^{[\frac{M-1}{2}]} \left|C_{2n+1}^{M}(2k_0)\right|^2 \tag{4}$$

$$|\lambda_0'|^{-2} = \sum_{s=1}^{[\frac{M}{2}]} \left|C_{2s}^{M}(2k_1 + 1)\right|^2 \tag{5}$$

$C_{2n+1}^{M}(2k_0)$ and $C_{2s}^{M}(2k_1 + 1)$ are the probability amplitudes of OEBS. Here n, s, and M are integers and η is generally complex with $0 \leq |\eta| \leq 1$.

Now we shall calculate the mean photon number $\overline{n}$, which is the expectation value of the number operator $\hat{n} = a^{\dagger}a$ with respect to the OEBS of Eqs. (3a) and (3b). It is easy to show that

(i) for even k

$$\langle \hat{n} \rangle_{\mathrm{o}} = |\lambda_0|^2 \sum_{n=0}^{[\frac{M-1}{2}]} (2n + 2k_0 + 1)\left|C_{2n+1}^{M}(2k_0)\right|^2 \tag{6a}$$

(ii) for odd k

$$\langle \hat{n} \rangle_{\mathrm{o}} = |\lambda_0'|^2 \sum_{s=0}^{[\frac{M}{2}]} (2s + 2k_1 + 1)\left|C_{2s}^{M}(2k_1 + 1)\right|^2 \tag{6b}$$

The expectation value of $\hat{n}^2$, namely ${}_{\mathrm{o}}\langle 2k_0, \eta, M|\hat{n}^2|2k_0, \eta, M\rangle_{\mathrm{o}}$ and ${}_{\mathrm{o}}\langle 2k_1 + 1, \eta, M|\hat{n}^2|2k_1 + 1, \eta, M\rangle_{\mathrm{o}}$ are given by

(i) for even k

$$\langle n^2 \rangle_{\mathrm{o}} = |\lambda_0|^2 \sum_{n=0}^{[\frac{M-1}{2}]} (2n + 2k_0 + 1)^2\left|C_{2n+1}^{M}(2k_0)\right|^2 \tag{7a}$$

(ii) for odd k

$$\langle n^2\rangle_o = |\lambda_0'|^2 \sum_{s=0}^{[\frac{M}{2}]} (2s + 2k_1 + 1)^2 \left|C_{2s}^M(2k_1+1)\right|^2 \tag{7b}$$

We shall employ the Glauber second-order correlation function to discuss some statistical properties such as sub-Poissonian distribution (Perina, 1984; Walls and Milburn, 1994), which is characteristic of nonclassical states. The Glauber second-order correlation function $g^{(2)}(0)$ is defined by

$$g^{(2)}(0) = \frac{\langle a^{\dagger 2} a^2\rangle}{\langle a^\dagger a\rangle^2} = \frac{\langle n^2\rangle - \langle n\rangle}{\langle n\rangle^2} \tag{8}$$

A light field has a sub-Poissonian distribution if $g^{(2)}(0) < 1$, which is a nonclassical effect; super-Poissonian distribution if $g^{(2)}(0) > 1$, which is a classical effect; and Poissonian distribution (characteristic of the coherent state) if $g^{(2)}(0) = 1$.

It is apparent from Eqs. (6) and (7) together with Eq. (8) that the value of $g^{(2)}(0)$ is always less than one for all values of M, k, and $|\eta|$, which signifies the sub-Poissonian statistics of the field in OEBS. In Fig. 1, we plot $g^{(2)}(0)$ of OEBS as a function of η for different values of M and k (η is taken to be real). It is to be noticed that as $\eta \to 0$, the first state (first odd number k) is present and the limiting value for the Fock state is obtained and the value of $g^{(2)}(0)$ increases as k increases. On the other hand, as $\eta \to 1$ with the increase in the parameter M, the function $g^{(2)}(0)$ approaches unity more rapidly and persists because the limiting Fock state in this case is the largest odd state less than or equal to $(M + k)$.

3. NORMAL SQUEEZING

The squeezing phenomenon represents one of the interesting phenomena in the field of quantum optics, and is a direct consequence of Heisenberg's uncertainty principle. It reflects the reduced quantum fluctuations in one of the field quadratures at the expense of stretching the other quadrature. The state is said to be squeezed if it has less noise than the vacuum state in one of the field quadratures. The investigation of normal squeezing is based on defining two field quadrature operators by

$$\hat{X} = \frac{1}{2}(a + a^\dagger) \quad \text{and} \quad \hat{Y} = \frac{1}{2i}(a - a^\dagger) \tag{9}$$

These operators satisfy the commutation relation

$$[\hat{X}, \hat{Y}] = \frac{i}{2} \tag{10}$$

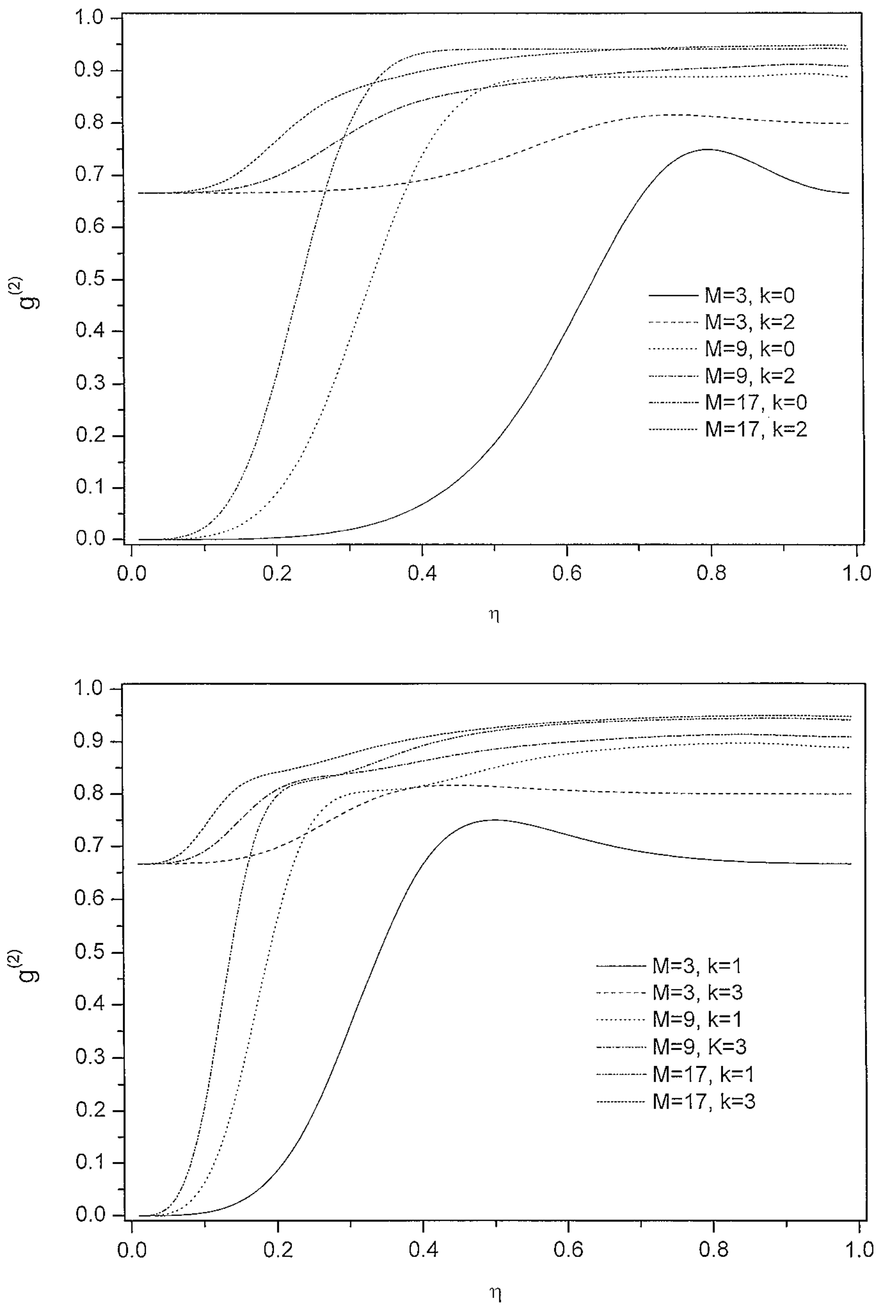

Fig. 1. $g^{(2)}$ parameter of the odd EBS as a function of η.

Therefore the uncertainty relation for the variances of $\hat{X}$ and $\hat{Y}$ is

$$(\Delta\hat{X})^2(\Delta\hat{Y})^2 \geq \frac{1}{16} \tag{11}$$

where the quadrature variances are

$$(\Delta\hat{X})^2 = \langle\hat{X}^2\rangle - \langle\hat{X}\rangle^2 \tag{12}$$

and

$$(\Delta\hat{Y})^2 = \langle\hat{Y}^2\rangle - \langle\hat{Y}\rangle^2 \tag{13}$$

The field is said to be squeezed if $\Delta^2 X$ or $\Delta^2 Y \leq \frac{1}{4}$.
Then normal squeezing holds if

$$S_1 = 4(\Delta\hat{X})^2 - 1 < 0 \tag{14a}$$

$$S_2 = 4(\Delta\hat{Y})^2 - 1 < 0 \tag{14b}$$

The squeezing parameters depend on the expectation values of the creation and annihilation operator ($a^\dagger$ and a) and their powers. From the definition of the OEBS (Eqs. (3) and (4)) it is abvious that the expectation values of operators a and $a^\dagger$ vanish. But the expectation values of operators a^2 and $a^{\dagger 2}$ are

(i) for even k

$$\begin{aligned}{}_{\mathrm{o}}\langle 2k_0, \eta, M|a^{\dagger 2}|2k_0, \eta, M\rangle_{\mathrm{o}} &= |\lambda_0|^2 \frac{\eta^{*2}}{(1-|\eta|^2)} \sum_{n=0}^{[\frac{M-2}{2}]} \left[\binom{M}{2n+1} \right. \\ &\times \frac{(2n+2k+3)!}{(2n+3)!} [(M-2n-1)(M-2n-2)]^{\frac{1}{2}} \\ &\left. \times (|\eta|^2)^{(2n+1)}(1-|\eta|^2)^{M-2n-1} \right]\end{aligned} \tag{15a}$$

(ii) for odd k

$$\begin{aligned}{}_{\mathrm{o}}\langle 2k_1+1, \eta, M|a^{\dagger 2}|2k_1+1, \eta, M\rangle_{\mathrm{o}} &= |\lambda_0'|^2 \frac{\eta^{*2}}{(1-|\eta|^2)} \sum_{s=0}^{[\frac{M-1}{2}]} \left[\binom{M}{2s} \right. \\ &\left. \times \frac{(2s+2k+2)!}{(2s+2)!} [(M-2s)(M-2s-1)]^{\frac{1}{2}} (|\eta|^2)^{(2s)}(1-|\eta|^2)^{M-2s} \right]\end{aligned} \tag{15b}$$

and $\langle a^{\dagger 2}\rangle = \langle a^2\rangle^*$.

We can study the squeezing effects of OEBS by combining Eqs. (3), (6), (14), and (15).

4. AMPLITUDE-SQUARED SQUEEZING (ASS)

The phenomenon of squeezing has been extended to higher order squeezing. The concept of higher order squeezing has been discussed by many authors (Hillery, 1987a,b, 1989; Hong and Mandel, 1985a,b; Perina, 1984; Walls and Milburn, 1994) and in this section we discuss the production of higher order squeezing in the sense of Hillery's definition (Hillery, 1987a,b). This type of squeezing is known as ASS and arises in a natural way in second-harmonic generation and in a number of nonlinear optical processes (Hillery, 1987a,b). This phenomenon is discussed through the fluctuations in the variances of the operators (Hillery, 1989)

$$\hat{d}_0 = \frac{1}{4}(aa^\dagger + a^\dagger a) \tag{16a}$$

$$\hat{d}_1 = \frac{1}{4}(a^2 + a^{\dagger 2}), \tag{16b}$$

$$\hat{d}_2 = \frac{1}{4i}(a^2 - a^{\dagger 2}) \tag{16c}$$

Operators $\hat{d}_1$ and $\hat{d}_2$ satisfy the commutation relation

$$[\hat{d}_1, \hat{d}_2] = i\hat{d}_0 \tag{17}$$

so that the uncertainty principle applied to $\hat{d}_1$ and $\hat{d}_2$ is

$$(\Delta\hat{d}_1)^2(\Delta\hat{d}_2)^2 \geq \frac{1}{4}\langle\hat{d}_0\rangle^2 \tag{18}$$

Therefore the condition for ASS is

$$q_1 = (\Delta\hat{d}_1)^2 - \frac{1}{2}|\langle\hat{d}_0\rangle| < 0 \tag{19a}$$

or

$$q_2 = (\Delta\hat{d}_2)^2 - \frac{1}{2}|\langle\hat{d}_0\rangle| < 0 \tag{19b}$$

The expectation value of the operator $a^{\dagger 4}$ can be seen to take the form

(i) for even k

$$\begin{aligned} {}_{\mathrm{o}}\langle 2k_0, \eta, M|a^{\dagger 4}|2k_0, \eta, M\rangle_{\mathrm{o}} &= |\lambda_0|^2 \frac{\eta^{*4}}{(1-|\eta|^2)^2} \sum_{n=0}^{[\frac{M-4}{2}]} \left[\binom{M}{2n+1} \right. \\ &\times \frac{(2n+2k+5)!}{(2n+5)!}[(M-2n-1)(M-2n-2)(M-2n-3) \\ &\left. \times (M-2n-4)]^{\frac{1}{2}} (|\eta|^2)^{2n+1}(1-|\eta|^2)^{M-2n-1} \right] \end{aligned} \tag{20a}$$

(ii) for odd k

$$
\begin{aligned}
{}_{\mathrm{o}}\langle 2k_1+1,\eta,M|a^{\dagger 4}|2k_1+1,\eta,M\rangle_{\mathrm{o}} &= |\lambda_0'|^2 \frac{\eta^{*4}}{(1-|\eta|^2)^2} \sum_{s=0}^{\left[\frac{M-3}{2}\right]} \Bigg[\binom{M}{2s} \\
&\quad \times \frac{(2s+2k+5)!}{(2s+4)!} [(M-2s)(M-2s-1)(M-2s-2) \\
&\quad \times (M-2s-3)]^{\frac{1}{2}} (|\eta|^2)^{2s} (1-|\eta|^2)^{M-2s} \Bigg]
\end{aligned}
\tag{20b}
$$

Combining Eqs. (6), (7), (19), and (20), we can study the squeezing effects of the OEBS. Numerical investigations show that no normal squeezing exists for OEBS when taking η real but the ASS is present. In Fig. 2 we plot the quantity q_1 of the OEBS as a function of the parameter η for different values of k and M. In these figures we see that as M increases, oscillations appear for small values of η; thus the behavior of the curves differs considerably from the odd BS ($k = 0$). Furthermore, we can see that for the same M the point of maximum squeezing shifts very slightly to lower η as k increases. However, as M increases, the degree of ASS increases.

5. QUASIPROBABILITY DISTRIBUTION FUNCTIONS

The quasiprobability distribution functions (Cahill and Glauber, 1969; Hillery *et al.*, 1984; Wigner, 1932) are important tools to discuss the statistical description of a microscopic system, and also to provide insight into the nonclassical features of the radiation field. There are three types of these functions: the P (Glauber–Sudershan) function, W (Wigner) function, and Q (Husimi) function.

In the present section we turn our attention to examine Q- and W-functions for the OEBS. It is well known that Q-function is positive definite at any point in the phase space for any quantum state. It can be written in an equivalent form as the expectation value of the density matrix $\hat{\rho}$ with respect to the coherent state $|\alpha\rangle$ as

$$
\langle\alpha|\hat{\rho}|\alpha\rangle = \pi Q(\alpha) \tag{21}
$$

where

$$
|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right) \sum_{\ell=0}^{\infty} \frac{\alpha^\ell}{\sqrt{\ell!}} |\ell\rangle \tag{22}
$$

($\alpha = x + iy$ is a complex number), and the density matrix in this case has the form

$$
\hat{\rho}_0 = |2k_0,\eta,M\rangle_{\mathrm{o}\,\mathrm{o}}\langle 2k_0,\eta,M| \quad \text{for even } k
$$

or

$$
\hat{\rho}_0 = |2k_1+1,\eta,M\rangle_{\mathrm{o}\,\mathrm{o}}\langle 2k_1+1,\eta,M| \quad \text{for odd } k
$$

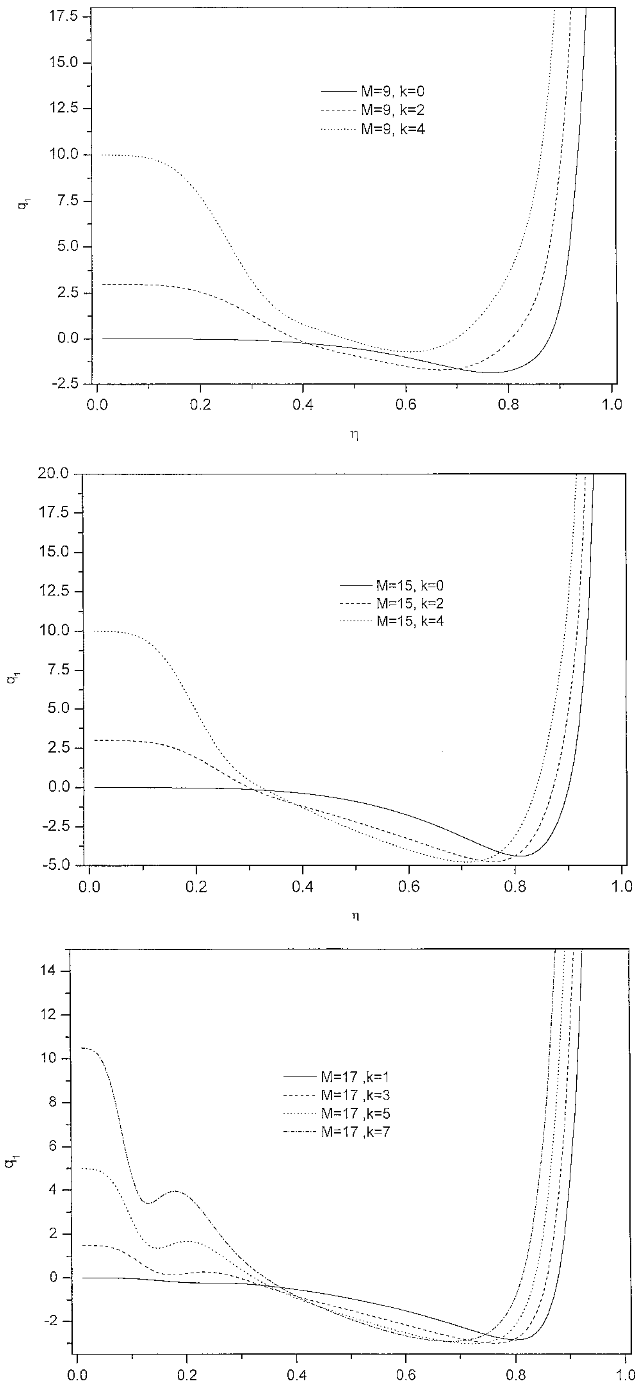

Fig. 2. The amplitude-squared squeezing for odd EBS.

Thus the Q-function is given by the following expressions:

(i) for even k

$$Q_0(\alpha) = \pi^{-1}|\lambda_0|^2\, e^{-|\alpha|^2}|G_0|^2 \tag{23a}$$

where

$$G_0 = \sum_{n=0}^{[\frac{M-1}{2}]}\left[\sqrt{\frac{M!}{(M-2n-1)!}}\,\frac{(\alpha^*)^{2n+2k_0+1}}{(2n+1)!}\eta^{2n+1}(1-|\eta^2|)^{\frac{M-2n}{2}}\right]$$

(ii) for odd k

$$Q_0(\alpha) = \pi^{-1}|\lambda_0'|^2\, e^{-|\alpha|^2}|G_1|^2 \tag{23b}$$

where

$$G_1 = \sum_{s=0}^{[\frac{M}{2}]}\left[\sqrt{\frac{M!}{(M-2s)!}}\,\frac{(\alpha^*)^{2s+2k_1+1}}{(2s)!}\eta^{2s}(1-|\eta|^2)^{\frac{M-2s}{2}}\right]$$

In Fig. 3 we have plotted the Q-function for different values of M, k, and η. We find that for small $\eta\,(\eta = 0.1)$ the first odd state $\leq k$ is the most effective one, as shown in Fig. 3(a). When η increases more states come to affect the Q-function. This means spreading out in the phase space and its diameter increases as the number of states increases as shown in Fig. 3(b) and (c). This can also be observed by increasing the value of the parameter M.

The W-function $W(\alpha)$ can take on negative values for some states and this is regarded as an indication of the nonclassical behavior. We shall take diagonal terms into account only in W-function which is given by

(i) for even k, i.e., $k = 2k_0$

$$W(\alpha) = -\frac{2}{\pi}|\lambda_0|^2\, e^{-2|\alpha|^2}\sum_{n=0}^{[\frac{M-1}{2}]}\left[\binom{M}{2n+1}(|\eta|^2)^{2n+1}\right.$$
$$\left.\times\ (1-|\eta|^2)^{M-2n-1}\frac{(2n+2k_0+1)!}{(2n+1)!}L_{2n+2k_0+1}(4|\alpha|^2)\right] \tag{24}$$

where $L_{2n+2k_0+1}(4|\alpha|^2)$ is Laguerre polynomials of order $(2n+2k_0+1)$, where

$$L_r(x) = \sum_{m=0}^{r}(-1)^m\frac{(r)!(x)^m}{(m!)^2(r-m)!}$$

(ii) for odd k, i.e., $k = 2k_1 + 1$

$$W(\alpha) = -\frac{2}{\pi}|\lambda_0'|^2\, e^{-2|\alpha|^2}\sum_{s=0}^{[\frac{M}{2}]}\left[\binom{M}{2n}(|\eta|^2)^{2s}\right.$$
$$\left.\times\ (1-|\eta|^2)^{M-2s}\frac{(2s+2k_1+1)!}{(2s)!}L_{2s+2k_1+1}(4|\alpha|^2)\right] \tag{25}$$

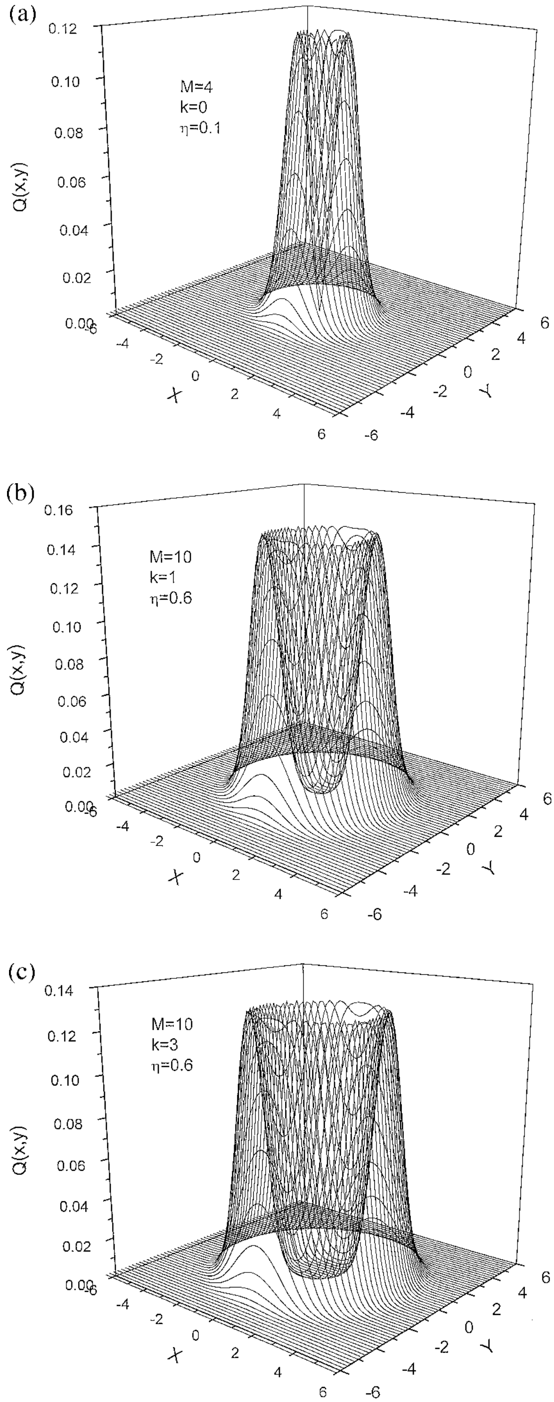

Fig. 3. *Q*-function corresponding to odd EBS.

To show the nonclassical properties we plot the function $W(\alpha)$ in Fig. 4 for different values of M, k, and η. When η is small ($\eta = 0.2$), $M = 5$, and $k = 0$, the $W(\alpha)$ for OEBS has a nonclassical character which appears clearly with large negative values, with a spike centered at the origion (see Fig. 4(a)). Increasing

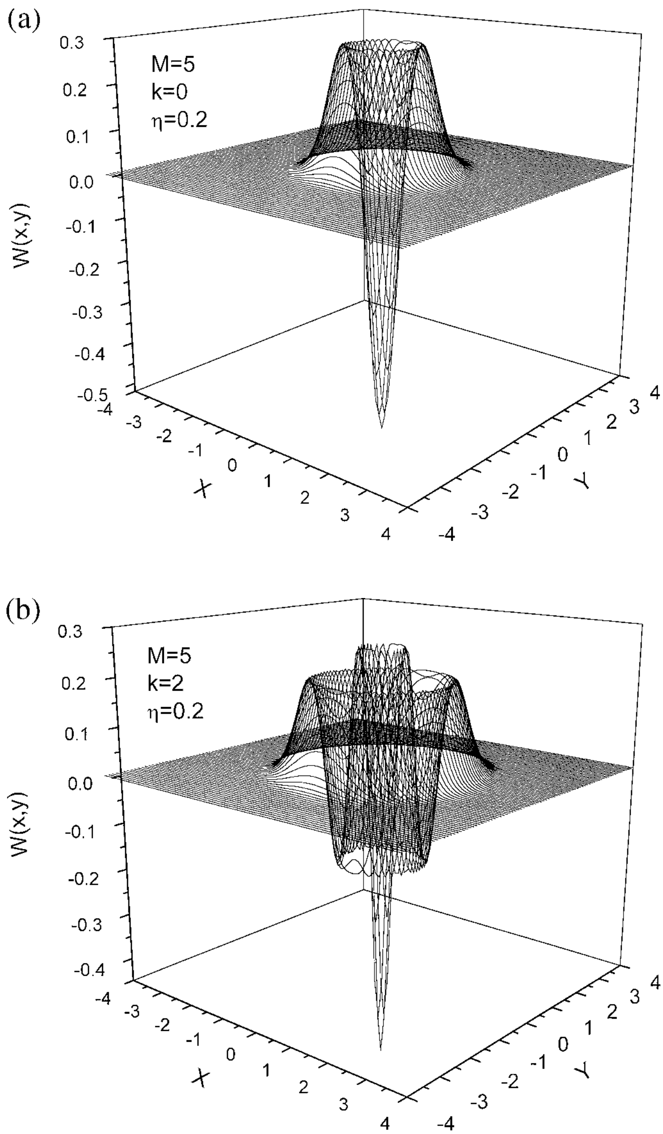

Fig. 4. Wigner function corresponding to odd EBS.

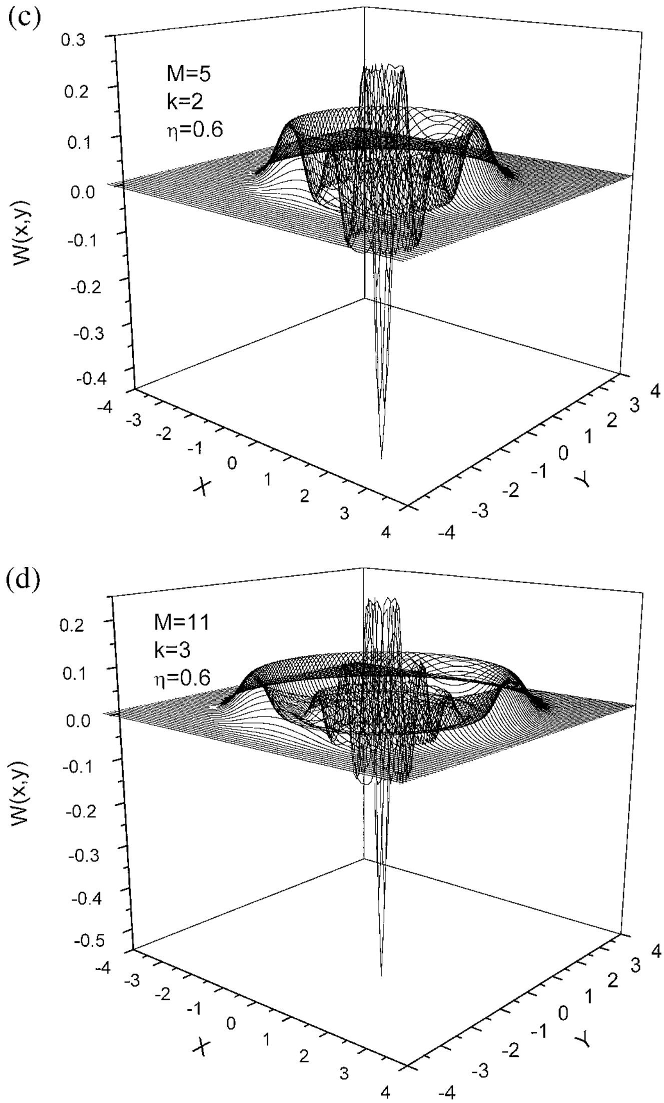

Fig. 4. (*Continued*)

k, very small oscillations surrounding the spice in the W-function are observed for the OEBS $|5, 2, 0.2>_{\text{odd}}$ in contrast to the odd BS (El-Orany *et al.*, 1999). However, increasing $|\eta|$ to 0.6, adds more wobbles (see Fig. 4(c)). As M increases ($M = 11$), we note a remarkable change in the shape of the function, as shown in Fig. 4(d).

6. CONCLUSION

In this paper, we have introduced the OEBS. Properties of these states are considered. It has been found that these states do not show normal squeezing, but ASS is exhibited for such states. The second correlation function is discussed and partial coherence is shown for various parameters. The quasi-distribution functions especially the W- and Q-functions are investigated and studied numerically for certain parameters. Nonclassical signatures for these states are present in the figures of the Wigner function in particular. These functions are not just theoretical curiosities, they can be detected in homodyne experiments (Leonhardt, 1997).

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